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# Random Fields: Skorohod integral and Malliavin derivative

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**Abstract**

We consider a general version of the Skorohod integral and of the Malliavin derivative with respect to the Lévy stochastic measures on general topological spaces. The integral operator is introduced as a limit of appropriate simple integrals in line with the classical integration schemes and it appears as a generalization of the original Skorohod integral. Moreover a direct differentiation formula is given for the derivative of the polynomials with respect to the Gaussian and the Poisson type stochastic measures.

Let $\mu(dx)$, $x \in X$, be a stochastic measure with independent values on a general separable topological space $X$ with a tight $\sigma$-finite Borel measure $\mathcal{M}(dx)$ having *no atoms*. Among these measures we consider the *Lévy stochastic measures* $\mu(dx)$, $x \in X$, which are characterized by the *Lévy-Khintchine infinitely divisible law* in the form:

$$\log Ee^{iu\mu(\Delta)} = \mathcal{M}(\Delta)\Big\{ -\frac{u^2\sigma^2}{2} + \int_{\mathbb{R}} \big(e^{iur} - 1 - iur\big)\mathcal{N}(dr)\Big\}, \quad u \in \mathbb{R}, \tag{1}$$

for the values $\mu(\Delta)$ on the Borel sets $\Delta \subseteq X$ such that $\mathcal{M}(\Delta) < \infty$. In the law above $\sigma^2 \geq 0$ is a constant and $\mathcal{N}(dr)$, $r \in \mathbb{R}$, is a $\sigma$-finite Borel measure on $\mathbb{R} \setminus \{0\}$.

The Skorohod integral and the Malliavin derivative are well-known with respect to the *Wiener stochastic measure* defined via the increments of the Wiener process. This measure would correpond to the case where $X = [0, \infty)$ and $\mathcal{N}(dr) \equiv 0$ in (1). We refer to [8]. See also [9] and [11], for example.

The goal of this paper is to introduce an *integral operator* and its ajoint, the *derivative operator*, with respect to a general Lévy stochastic measure such that the integral is constructed as limit of appropriate simple integrals and appears as a generalization of the Skorohod integral, while its ajoint appears as a generalization of the Malliavin derivative.

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We consider the following scheme of integration. Let $L_2(\Omega)$ be the Hilbert space of real random variables $\xi := \xi(\omega)$, $\omega \in \Omega$, with the norm $\|\xi\| = \left(E|\xi|^2\right)^{1/2}$, and let $H \subseteq L_2(\Omega)$ be the closure in $L_2(\Omega)$ of all the polynomials

$$\xi = F(\xi_1, ...\xi_m) \quad \text{with} \quad \frac{\partial^k}{\partial \xi_j^k} F(\xi_1, ...\xi_m) = 0, \quad k \neq 1 \quad (j = 1, ..., m), \tag{2}$$

of the values $\xi_j := \mu(\Delta_j)$ of the stochastic measure $\mu(dx)$, $x \in X$, on the *disjoint* Borel sets $\Delta_j$, $j = 1, ..., m$. In other words the above polynomials are *multilinear forms* of the variables $\xi_j$, $j = 1, ..., m$. Let $L_2(\Omega \times X)$ be the standard space of real stochastic functions $\varphi := \varphi(x)$, $x \in X$, in $L_2(\Omega)$ with the norm $\|\varphi\|_{L_2} = \Big( \int_X \|\varphi(x)\|^2 \mathcal{M}(dx) \Big)^{1/2}$, and let $L_2(X, H) \subseteq L_2(\Omega \times X)$ be the subspace of all the stochastic functions taking values in the subspace $H \subseteq L_2(\Omega)$. Here we focus, in particular, on the linear class $L \subseteq L_2(X, H)$ of *simple functions* admitting representation

$$\varphi := \sum_{\Delta \subseteq X} \varphi_\Delta 1_\Delta(x), \qquad x \in X, \tag{3}$$

via some appropriate *disjoint* sets $\Delta \subseteq X$. For all $\Delta$, the value $\varphi_\Delta$, taken on the corresponding $\Delta$, is a random variable measurable with respect to the $\sigma$-algebra $\mathfrak{A}_{]\Delta[}$ which is generated by the values of the stochastic measure $\mu(dx)$, $x \in X$, on all the subsets of the complement $]\Delta[$ to $\Delta$ in $X$, i.e. $]\Delta[ := X \setminus \Delta$.

**Theorem 1.** *The closable linear operator* $I : L_2(X, H) \supseteq L \ni \varphi \Longrightarrow I\varphi \in H$, *is well defined on the simple functions of the type* (3) *as* $I\varphi := \sum_{\Delta \subseteq X} \varphi_\Delta \mu(\Delta)$. *Its minimal closed linear extension*

$$I : \quad L_2(X, H) \supseteq domI \ni \varphi \quad \Longrightarrow I\varphi \in H \tag{4}$$

*defines the* stochastic integral $I\varphi := \int_X \varphi(x)\mu(dx)$. *All the elements* $\xi \in H$ *can be represented as* $\xi = I\varphi$ *by means of the corresponding integrands* $\varphi$ *in the domain* $domI$ *of the operator* (4) *which is dense in* $L_2(X, H)$.
*For the ajoint (closed) linear operator, which we call* stochastic derivative,

$$D = I^* : \quad H \supseteq domD \ni \xi \quad \Longrightarrow \quad D\xi \in L_2(X, H), \tag{5}$$

*the duality relationship*

$$I = D^*, \quad D = I^* \tag{6}$$

*holds true.*

*Proof.* For $n = 1, 2, ...$, let $\Delta_{nk}$, $k = 1, ..., K_n$, be some *disjoint* sets in $X$ with $\lim_{n\to\infty} \max_k \mathcal{M}(\Delta_{nk}) = 0$, constituting the $n^{th}$-series in an array $(n = 1, ...)$. The elements of each series are partition sets of all the sets of all the preceeding series and the whole family of the sets of all the series constitute a semi-ring generating the Borel $\sigma$-algebra in $X = \lim_{n\to\infty} \sum_{k=1}^{K_n} \Delta_{nk}$. Let us consider, for any particular $m = 1, 2, ...$ and any choice of *different* elements $\Delta_{nk_j}$, $j = 1, ..., m$, in the *same* $n^{th}$-series, the corresponding $m$-order *multilinear form*, i.e.

$$\xi := \prod_{j=1}^{m} \mu(\Delta_{nk_j})$$

- cf. (2). These multilinear forms are *orthogonal* in $L_2(\Omega)$, see [4]. Let $H^{n,m}$ be the subspace in $L_2(\Omega)$ with the orthogonal basis of all the above multilinear forms. And let $L_2^{n,m-1}$ be the subspace in the functional space $L_2(X, H)$ of the simple functions which admit representation (3) via the sets $\Delta = \Delta_{nk_j}$, $j = 1, ..., m$, belonging to the same $n^{th}$-series above. The corresponding values $\varphi_\Delta$ belong to $H^{n,m-1}$. Here we set $H^{n,0}$ to be the subspace of all *constants* in $L_2(\Omega)$.
The linear operator $D = I^*$: $H^{n,m} \ni \xi \implies D\xi \in L_2^{n,m-1}$, takes the values

$$D\big(\prod_{j=1}^{m} \mu(\Delta_{nk_j})\big) := \sum_{j=1}^{m} \big(\prod_{i\neq j} \mu(\Delta_{nk_i})\big) 1_{\Delta_{nk_j}},$$

on the elements of the orthogonal basis in $H^{n,m}$ and it is such that $\|D\xi\|_{L_2} = m^{1/2}\|\xi\|$, $\xi \in H^{n,m}$.
The simple functions of the form

$$\varphi := \varphi_\Delta 1_\Delta(x),\ x \in X, \quad \text{with} \quad \varphi_\Delta := \prod_{j=1}^{m-1} \mu(\Delta_{nk_j}) \quad \text{and} \quad \Delta := \Delta_{nk_m},$$

constitute a basis in $L_2^{n,m-1}$. The orthogonal projection of these functions on the range of the operator $D$, i.e. $DH^{n,m} \subseteq L_2^{n,m-1}$, is given by $\hat{\varphi} := m^{-1} D\big(\prod_{j=1}^{m} \mu(\Delta_{nk_j})\big)$. The *adjoint* operator $D^*$ gives

$$D^*\hat{\varphi} = \prod_{j=1}^{m} \mu(\Delta_{nk_j}) = \varphi_\Delta \mu(\Delta),$$

since the operator $m^{-1/2}D$ is *isometric*. Recall that $I\big(\varphi_\Delta 1_\Delta\big) = \varphi_\Delta \mu(\Delta)$ holds and moreover note that the operator $I$ is null on the orthogonal complement to $DH^{n,m}$ in $L_2^{n,m-1}$. In fact it is $I\big(\varphi - \hat{\varphi}\big) = I\varphi - I\hat{\varphi} = 0$. Thus we have $D^*\varphi = I\varphi$, for all the elements $\varphi$ of the orthogonal basis in $L_2^{n,m-1}$. Exploiting the limits

$$H = \lim_{n\to\infty} \sum_{m=0}^{\infty} \oplus H^{n,m}, \quad L_2(X, H) = \lim_{n\to\infty} \sum_{m=1}^{\infty} \oplus L_2^{n,m-1},$$

we can complete the proof. $\square$

In the sequel we refer to the *Gaussian stochastic measure* as the measure $\mu(dx)$ characterized by the law (1) with $\mathcal{N}(dr) \equiv 0$ and we refer to the *stochastic measure of the Poisson* *type* for $\mu(dx)$ characterized by (1) with $\sigma^2 = 0$ and $\mathcal{N}(dr)$ concentrated in some single point $\rho \neq 0$. Recall that in the latter case it is $\mu(\Delta) := \rho\big[\nu(\Delta) - E\nu(\Delta)\big]$ for the Poisson variables $\nu(\Delta)$, $\Delta \subseteq X$.

**Theorem 2.** *(i) The* stochastic derivative $D$ *in* (5) *is the minimal closure of the closable linear operator well-defined on all the multilinear forms* (2) *by the differetiation formula*

$$D\xi := \sum_{j=1}^{m} \frac{\partial}{\partial \xi_j} F(\xi_1, ..., \xi_m) 1_{\Delta_j}(x), \quad x \in X. \tag{7}$$

*(ii) The* family of all the polynomials $\xi$ *of the values* $\mu(\Delta)$, $\Delta \subseteq X$, *belongs to the domain of the Malliavin derivative if and only if the Lévy stochastic measure* $\mu(dx)$, $x \in X$, *is either Gaussian or of the Poisson type.*
*(iii) In the latter case, the stochastic derivative of* any *polynomial* $\xi$, *given in the representation* $\xi = F(\xi_1, ..., \xi_m)$ *via the values* $\xi_j = \mu(\Delta_j)$ *on the* disjoint *sets* $\Delta_j$, $j = 1, ..., m$, *can be computed by the following differentiation formula*

$$D\xi = \sum_{j=1}^{m} \Big[\frac{\partial}{\partial \xi_j} F(\xi_1, ..., \xi_m) + \sum_{k \geq 2} \frac{\rho^{k-1}}{k!} \frac{\partial^k}{\partial \xi_j^k} F(\xi_1, ..., \xi_m)\Big] 1_{\Delta_j}(x), \quad x \in X. \tag{8}$$

*(iv) This above formula* (7) *is also valid in the Gaussian case by setting* $\rho = 0$.

*Proof.* For this we refer to [4]. Here, we would like only to note that the relationships

$$H = L_2(\Omega), \quad L_2(X, H) = L_2(\Omega \times X) \tag{9}$$

hold true *only* if and only if the Lévy stochastic measures involved are Gaussian or of the Poisson type.

**Remark.** The operator $D$ in (5) represents the analogue of the *Malliavin derivative* and the operator $I$ in (4) which is also $I = D^*$ - cf. (6), represents the corresponding analogue of the *Skorohod integral* - cf. [13]. Thus we can call them *Malliavin derivative* and *Skorohod integral* correspondingly. Here we refer to e.g. [8], [9], [11]. We also stress that recently there has been a wide activity on the generalization of the Malliavin calculus to stochastic measures of the Lévy type such as measures derived from Poisson processes (i.e. $X = [0, \infty)$ and $\sigma = 0$, $\mathcal{N}(dx)$ concentrated in 1, in the law (1)), from pure jump Lévy processes (i.e. $X = [0, \infty)$ and $\sigma = 0$ in (1)), or from more general Lévy processes on $X = [0, \infty)$. Not being allowed to an exhaustive list, we might mention e.g. [1], [2], [3], [5], [6], [7], [10], [12] [14], and the references therein.

**Question.** For a general Lévy stochastic measure $\mu(dx)$, $x \in X$, the Malliavin derivative of the basic multilinear forms (2) can be determined as the limit

$$D\xi = \lim_{n \to \infty} \sum_{k=1}^{K_n} E\Big(\xi \frac{\mu(\Delta_{nk})}{\|\mu(\Delta_{nk})\|^2} |\mathfrak{A}_{]\Delta_{nk}[}\Big) 1_{\Delta_{nk}}(x), \quad x \in X, \tag{10}$$

in $L_2(X,H)$, via the array of series of disjoint sets $\Delta_{nk}$, $k = 1, ..., K_n$ $(n = 1, 2, ...)$ - cf. [4]. And here an open question is whether this differentiation formula (10) holds true for all $\xi \in domD$.

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